Multiplications of Maximal Rank in the Cohomology of $\mathbb{P}^1 \times \mathbb{P}^1$

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Abstract

Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and let $C \subseteq Q$ be a curve of type (a, b) of equation $F = 0$. The main purpose of this paper is to analize the multiplicative structure of the bi-graded module $H^1_* \mathscr{O}_Q$, in particular to prove that for any $r, s \geq 0$ the multiplication map $H^1 \mathscr{O}_Q(r, -s) \xrightarrow{F} \mathscr{O}_Q(r + a, -s + b)$ induced by F has maximal rank for the general C of type (a, b). Interpretations of this problem in the contexts of multilinear algebra and differential algebra are emphasized.

1 Introduction.

Tensors enter in many fields of pure and applied mathematics and the most common and useful linear maps on spaces of tensors arise from multiplications and contractions. In this paper we address a quite natural question about multiplication and contraction maps of symmetric tensors. We work over the field of the complex numbers $\mathbb{C}$ and set $V = \mathbb{C}^{m+1}$, $W = \mathbb{C}^{n+1}$. Denote symmetric powers with S^i.

Problem 1.1. *Let $\sigma \in S^a V \otimes S^b W$ be some fixed element and r, t be some integers with $r \geq 0$ and $t \geq b$. Consider the linear map*

$$S^r V \otimes S^t W^* \xrightarrow{\sigma} S^{r+a} V \otimes S^{t-b} W^* \tag{1}$$

defined by multiplication on the first r tensor components and contraction on the last t tensor components. Is this map of maximal rank, for σ sufficiently general in $S^a V \otimes S^b W$?

Notice that the question is very easy if $\dim W$ or $\dim V = 1$, indeed in these cases the map (1) is given by multiplication of symmetric tensors or by its dual, the contraction, and it is either injective or surjective. The first non-trivial case of Problem 1.1 appears when $\dim V = \dim W = 2$. The

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object of this paper is to solve Problem 1.1 in this case. We like also to point out two other equivalent formulations of Problem 1.1. Consider the variables $\underline{x} = (x_0, \ldots, x_m)$, $\underline{y} = (y_0, \ldots, y_n)$ and the derivations $\underline{\partial} = (\partial y_0, \ldots, \partial y_n)$. We denote with $\mathbb{C}[\underline{x}]_i$ the vector space of homogeneous polynomials of degree i, for any $i \geq 0$.

Problem 1.2. *Consider a differential operator $D \in \mathbb{C}[\underline{x}]_a \otimes \mathbb{C}[\underline{\partial}]_b$ and the linear map*

$$D : \mathbb{C}[\underline{x}]_r \otimes \mathbb{C}[\underline{y}]_t \to \mathbb{C}[\underline{x}]_{r+a} \otimes \mathbb{C}[\underline{y}]_{t-b}. \qquad (2)$$

Is this map of maximal rank if D is sufficiently general in $\mathbb{C}[\underline{x}]_a \otimes \mathbb{C}[\underline{\partial}]_b$?

Now let $\mathbb{P}^m$ and $\mathbb{P}^n$ be projective spaces over $\mathbb{C}$ of dimensions m, n, respectively, $Q = \mathbb{P}^m \times \mathbb{P}^n$ their product and π_1, π_2 the first and second projection, respectively. Recall that $\mathrm{Pic}(Q) \cong \mathbb{Z} \times \mathbb{Z}$, with basis $\mathscr{O}_Q(1,0) = \pi_1^* \mathscr{O}_{\mathbb{P}^m}(1)$ and $\mathscr{O}_Q(0,1) = \pi_2^* \mathscr{O}_{\mathbb{P}^n}(1)$. In these notations, one may consider the following third version of Problem 1.1.

Problem 1.3. *Consider the multiplication map*

$$H^n \mathscr{O}_Q(r, -t-n-1) \xrightarrow{\sigma} H^n \mathscr{O}_Q(r+a, -t+b-n-1), \quad t \geq b \qquad (3)$$

with $\sigma \in H^0 \mathscr{O}_Q(a,b)$ a form of bi-degree (a,b). Is this map of maximal rank if σ is sufficiently general in $H^0 \mathscr{O}_Q(a,b)$?

The fact that the three problems above are equivalent is well known. For instance the equivalence of Problem 1.1 and Problem 1.3 is due to the fact that $H^n \mathscr{O}_Q(r, -t-n-1) = H^0 \mathscr{O}_{\mathbb{P}^m}(r) \otimes H^n \mathscr{O}_{\mathbb{P}^n}(-t-n-1) \cong S^r V \otimes S^t W^*$, for $V = H^0 \mathscr{O}_{\mathbb{P}^m}(1)$ and $W = H^0 \mathscr{O}_{\mathbb{P}^n}(1)$, by Künneth formula and Serre duality. Moreover the multiplication $H^n \mathscr{O}_{\mathbb{P}^n}(-t-n-1) \xrightarrow{\tau} H^n \mathscr{O}_{\mathbb{P}^n}(-t+b-n-1)$, with $\tau \in H^0 \mathscr{O}_{\mathbb{P}^n}(b)$, is dual to the multiplication $H^0 \mathscr{O}_{\mathbb{P}^n}(t-b) \xrightarrow{\tau} H^0 \mathscr{O}_{\mathbb{P}^n}(t)$. Denoting by $(y^*)^I$ the dual basis of $y^I = y_0^{i_0} \cdots y_n^{i_n}$, with $i_0 + \cdots + i_n = |I| = t$, this map can be described as follows. For any multi-index J with $|J| = b$, one has $(y^*)^I \cdot y^J = 0$ if $J \not\subset I$ and $(y^*)^I \cdot y^J = (y^*)^{I \setminus J}$ if $J \subset I$. This coincides with the differentiation $D(y^*)^I$, where $D = \partial_{y_0^*}^{i_0} \cdots \partial_{y_n^*}^{i_n}$, up to a non zero rational number factor, showing the equivalence of Problems 1.1 and 1.3 with Problem 1.2. In this paper we answer affirmatively to Problem 1.3 for $m = n = 1$, using some deep facts about the geometry of curves on the surface $Q = \mathbb{P}^1 \times \mathbb{P}^1$. We know also how to solve the three problems in case $a = b$ and n, m general, by a very different technique involving the differential operator formulation of Problem 1.2. The full solution of Problems 1.1,1.2,1.3 will be the object of future investigations. Notice that the fact that the map (3) has maximal rank in the case of $n = m = 1$ helps understanding the

multiplicative stucture of the bigraded module $H^1_* \mathscr{O}_Q$, and hence that of the Rao module of curves $C \subset Q$ when Q is embedded in $\mathbb{P}^3$, cfr. [2].

Aknowledgements. The present paper is a development of the previous paper [3] of the same authors. The first two authors acknowledge that its completion is mainly due to the contributions of the third author.

2 Problem 3 for $m = n = 1$. Notations and first reductions.

Let $F \in H^0 \mathscr{O}_Q(a, b)$, with $a, b \geq 1$ be a form of bi-degree (a, b) and let C denote the effective divisor associated to F; we call it a curve of type (a, b). We want to show that if C is sufficiently general in the complete linear system $|\mathscr{O}_Q(a, b)|$, then the multiplication map induced by F

$$H^1 \mathscr{O}_Q(r, -t-2) \xrightarrow{F} H^1 \mathscr{O}_Q(r+a, -t+b-2) \tag{4}$$

has maximal rank. Consider the exact sequence of sheaves

$$0 \to \mathscr{O}_Q(r, -t-2) \xrightarrow{F} \mathscr{O}_Q(r+a, -t+b-2) \to \mathscr{O}_C(r+a, -t+b-2) \to 0$$

naturally associated to F. Since $r \geq 0$ and $t \geq b$, one easily sees that $\ker(F) \cong H^0 \mathscr{O}_C(r+a, -t+b-2)$ and $\mathrm{coker}(F) \cong H^1 \mathscr{O}_C(r+a, -t+b-2)$. So Problem 3 for $m = n = 1$ will be equivalent to the following theorem, whose proof will occupy the remainder of this paper.

Theorem 2.1. *If $(a, b) \geq (1, 1)$ and $C \in |\mathscr{O}_Q(a, b)|$ is a general curve, then for any $h \geq a$ and $k \leq -2$ one has*

$$h^0 \mathscr{O}_C(h, k) \cdot h^1 \mathscr{O}_C(h, k) = 0. \tag{5}$$

Remark 2.2. Notice that the formulation of Problem 1.3 for $m = n = 1$ in the form of Theorem 2.1 above nicely illustrates its non-triviality. For example one can easily see that any curve of type (a, b) which contains a line of type $(0, 1)$ fails to verify (5). Even assuming smoothness, it is not difficult to find curves not verifying the statement of Theorem 2.1 for infinitely many (h, k). Consider for instance a smooth curve E of type $(2, 2)$ in Q, hence with $g(E) = 1$, with $\mathscr{O}_E(n, 0) \cong \mathscr{O}_E(0, n)$ for some $n \geq 2$. It is not difficult to show that such curves exist. Then one sees that $h^0 \mathscr{O}_E(ln, -ln) = \mathscr{O}_E$ for any $l \geq 0$, so (5) is false for $(h, k) = (ln, -ln)$.

The statement of Theorem 2.1 was proposed in our paper [3] as Conjecture 8.1. Notice that the cases $a = 1$ or $b = 1$ are trivial, since in these cases a general C of type (a, b) is isomorphic to $\mathbb{P}^1$ and the statement above is trivially verified, since for any line bundle L on $\mathbb{P}^1$ one has $h^0(L)h^1(L) = 0$. So from now on we assume $(a, b) \geq (2, 2)$. Set $d = \deg \mathscr{O}_C(h, k)$ and $g = (a - 1)(b - 1) = p_a(C)$. What we have to show is that if $d + 1 - g \leq 0$ then $h^0 \mathscr{O}_C(h, k) = 0$, or if $d + 1 - g \geq 0$ then $h^1 \mathscr{O}_C(h, k) = 0$. These two problems are equivalent. Indeed one has $\omega_C = \mathscr{O}_C(a - 2, b - 2)$ for any curve of type (a, b) on Q, hence $\omega_C \otimes \mathscr{O}_C(-h, -k) = \mathscr{O}_C(a - 2 - h, b - 2 - k)$. Setting $h' = a - 2 - h$ and $k' = b - 2 - k$ by the hypothesis $h \geq a$, $k \leq -2$ we find $h' \leq -2$ and $k' \geq b$, moreover $h^0 \mathscr{O}_C(h, k) \cdot h^1 \mathscr{O}_C(h, k) = h^0 \mathscr{O}_C(h', k') \cdot h^1 \mathscr{O}_C(h', k')$ hence the problem for h, k is equivalent to the one for h', k', up to the automorphism of $Q = \mathbb{P}^1 \times \mathbb{P}^1$ that interchanges the two rulings. An important final observation is that one only needs to construct a single curve satisfying the statement of Theorem 2.1, since, by semi-continuity, the statement will then hold also on a Zariski open subset of $|\mathscr{O}_Q(a, b)|$.

Further numerical reductions for (h, k).

The proof of Theorem 2.1 has been reduced to show that if $d + 1 - g \leq 0$ then $h^0 \mathscr{O}_C(h, k) = 0$. Notice that if $(h, k) \leq (\bar{h}, \bar{k})$ then there exists some sheaf embedding $\mathscr{O}_C(h, k) \subset \mathscr{O}_C(\bar{h}, \bar{k})$. So, if $\bar{d} = \deg \mathscr{O}_C(\bar{h}, \bar{k})$ satisfies $\bar{d} + 1 - g \leq 0$, and one is able to show that $h^0 \mathscr{O}_C(\bar{h}, \bar{k}) = 0$, one immediately deduces also $h^0 \mathscr{O}_C(h, k) = 0$. So we assume that this reduction is no more possible, that is $d \leq g - 1$, $\deg \mathscr{O}_C(h + 1, k) > g - 1$ and $\deg \mathscr{O}_C(h, k + 1) > g - 1$. This implies

$$g - 1 - b < d \leq g - 1$$
$$g - 1 - a < d \leq g - 1,$$

that is $g - 1 \geq d > ab - a - b - \min(a, b)$. Now let us write

$$h = \alpha + ma \quad \text{with } -1 \leq \alpha \leq a - 2$$
$$k = \beta - nb \quad \text{with } -1 \leq \beta \leq b - 2$$

Notice that $m, n > 0$, due to the limitations for h, k. Assume that $(\alpha, \beta) \neq (-1, -1)$. We want to show that in this case one has $m = n$. We have $d = \alpha b + \beta a + (m - n)ab \leq g - 1 = ab - a - b$. Since $(\alpha, \beta) > (-1, -1)$, one sees easily that $m \leq n$. If it were $m < n$, then $d \leq \alpha b + \beta a - ab \leq (a - 2)b + (b - 2)a - ab = ab - 2a - 2b < ab - a - b - \min(a, b)$, which is against our assumptions on d. So we have $m = n$. We have proved the following proposition.

Proposition 2.3. *Under the assumptions above, if $(\alpha, \beta) > (-1, -1)$ we have $\mathscr{O}_C(h, k) = \mathscr{O}_C(\alpha + ma, \beta - mb)$.*

If $\alpha = \beta = -1$ we have $\mathscr{O}_C(h, k) = \mathscr{O}_C(-1 + ma, -1 - nb)$ and $d = -b + mab - a - nab \le g - 1 = ab - a - b$, so $m - n \le 1$. Moreover $d > ab - a - b - \min(a, b)$, so $(m-n)ab > ab - \min(a, b)$, hence $m - n > 0$. So in the case $\alpha = \beta = -1$ we have $m = n+1$ and $\mathscr{O}_C(h, k) = \mathscr{O}_C(-1+ma, b-1-mb)$. We have proved the following.

Proposition 2.4. *Under the assumptions above, if $(\alpha, \beta) = (-1, -1)$ we have $\mathscr{O}_C(h, k) = \mathscr{O}_C(-1 + ma, b - 1 - mb)$.*

In the next section we will prove Theorem 2.1 in this last case by considering a very particular class of curves of type (a, b).

A special class of (a, b) curves

Take any a pairwise distinct $(1, 0)$ lines with equations $L_1 = 0, \ldots, L_a = 0$ and b pairwise distinct $(0, 1)$ lines with equations $M_1 = 0, \ldots, M_b = 0$. We denote G the grid formed by the points P_{ij} determined by the equations $L_i = M_j = 0$, for $i = 1, \ldots, a$ and $j = 1, \ldots, b$. Then the following result holds.

Lemma 2.5. *The general (a, b) curve C_0 containing G is smooth and has the property*

$$\mathscr{O}_{C_0}(a, -b) \cong \mathscr{O}_{C_0}.$$

Proof. Consider the $(a, 0)$ curve $L = L_1 \cdots L_a$ and the $(0, b)$ curve $M = M_1 \cdots M_b$. We can cover Q with charts (u, v) parametrizing affine open sets $U \cong \mathbb{A}^1 \times \mathbb{A}^1 \subset Q$, in such a way that $L = (u - \lambda_1) \cdots (u - \lambda_a) = l(u)$ and $M = (v - \mu_1) \cdots (v - \mu_b) = m(v)$ on U, with all λ's and μ's not equal to 0. Then we consider a curve C of type (a, b), containing G, with equation in U of the form

$$l(u)v^b - h(u)m(v) = 0, \tag{6}$$

with $\deg h(u) = a$. Then a singular point of C in U must satisfy $l(u)v^b - m(v)h(u) = l'(u)v^b - m(v)h'(u) = bl(u)v^{b-1} - m'(v)h(u) = 0$. One can exclude solutions with $v = 0$ by choosing $h(u)$ without multiple roots. Similarly, solutions with $m(v) = 0$ are impossible since $l(u)$ has no multiple roots. Choosing $h(u)$ such that $l(u)$ and $h(u)$ have no common roots, we see that any solution (u_0, v_0) of the system above must be such that the two vectors (v_0^b, bv_0^{b-1}) and $(m(v_0), m'(v_0))$ must be linearly dependent, hence $v_0^b m'(v_0) - bv_0^{b-1}m(v_0) = 0$. So v_0 varies in a specified finite set F. Moreover u_0 must be

a common root of $l(u)v_0^b - m(v_0)h(u) = 0$ and $l'(u)v_0^b - m(v_0)h'(u) = 0$, that is a multiple root of $l(u)v_0^b - m(v_0)h(u) = 0$. One can exclude this possibility by choosing $h(u)$ so that $l(u)c^b - m(c)h(u) = 0$ has no multiple roots for any $c \in F$. So there exists C of the form (6) smooth on U. Hence the general curve of type (a, b) containing G is smooth on U, and since we can cover Q with four such open affines U, we see such a general C is smooth everywhere. Finally we see that if C_0 is smooth and contains G, one has $\mathcal{O}_{C_0}(a, -b) \cong \mathcal{O}_{C_0}(C_0.L - C_0.M) = \mathcal{O}_{C_0}(G - G) = \mathcal{O}_{C_0}$. $\qquad\square$

Corollary 2.6. *The conclusion of Theorem 2.1 holds for*

$$\mathcal{O}_C(h, k) = \mathcal{O}_C(-1 + ma, b - 1 - mb).$$

Proof. Let C_0 be an (a, b)-curve as in the lemma above. Then $\mathcal{O}_{C_0}(a, -b) = \mathcal{O}_{C_0}$, whence $\mathcal{O}_{C_0}(h, k) = \mathcal{O}_{C_0}(-1, b - 1)$. Then one has $H^0\mathcal{O}_{C_0}(-1, b - 1) = \ker(H^1\mathcal{O}_Q(-1 - a, -1) \xrightarrow{F} H^1\mathcal{O}_Q(-1, b - 1)) = (0)$. $\qquad\square$

Now we are left with $\mathcal{O}_C(h, k)$ as in Proposition 2.3. This will be the object of the remaining sections.

3 Completion of the proof of Theorem 2.1.

Given α and β as in Proposition 2.3, we set $\hat{\alpha} = a - 2 - \alpha$ and $\hat{\beta} = b - 2 - \beta$. Notice that $\mathcal{O}_C(\hat{\alpha}, \hat{\beta}) = \omega_C(-\alpha, -\beta)$.

Let G be the grid of ab points in Q introduced in the preceding section. We will need the following technical result on the existence of a subset of G with particularly good properties for our purposes.

Lemma 3.1. *There exists a subset $Z \subset G$ such that $\deg Z = (\hat{\alpha} + 1)(\hat{\beta} + 1)$ and $H^0\mathscr{I}_Z(\alpha, \beta) = H^0\mathscr{I}_Z(\hat{\alpha}, \hat{\beta}) = 0$.*

To prove this fact, we need the following combinatorial result.

Lemma 3.2. *For any fixed positive integers r, l, N with $N \leq rl$, there exists a bipartite graph g with r right vertices and l left vertices, such that every right vertex has degree $\geq \lfloor N/r \rfloor$ and every left vertex has degree $\geq \lfloor N/l \rfloor$.*

Proof. The statement is trivial if $N = 1$ or if $N = rl$. One proceeds by induction on N. Denoting $(v_1, \ldots, v_r)$ and $(w_1, \ldots, w_l)$ the distribution of degrees at right and at left, respectively, with $\sum v_i = \sum w_j = N$, the statement consists in producing a graph with distributions of the form

$$
\begin{aligned}
\underline{v} &= (v_1, \ldots, v_r) = (v + 1, \ldots, v + 1, v \ldots, v) \\
\underline{w} &= (w_1, \ldots, w_l) = (w + 1, \ldots, w + 1, w \ldots, w).
\end{aligned}
$$

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Indeed in this case one has necessarily $v = \lfloor N/r \rfloor$ and $w = \lfloor N/l \rfloor$. Now the easy proof is left to the reader. $\qquad\square$

Proof of Lemma 3.1. We set $\gamma = \max(\alpha, \hat{\alpha})$ and $\delta = \max(\beta, \hat{\beta})$. Consider the sub-grid $G_{\gamma,\delta} = \{P_{i,j} : 1 \leq i \leq \gamma+1, 1 \leq j \leq \delta+1\}$. We will construct the required set Z as a subset of $G_{\gamma,\delta}$. Since we know that $(\alpha+1)(\beta+1) \leq (\hat{\alpha}+1)(\hat{\beta}+1)$, there are three possibilities: (γ, δ) equal to $(\hat{\alpha}, \hat{\beta})$ or to $(\hat{\alpha}, \beta)$ or to $(\alpha, \hat{\beta})$. In the first case we take $Z = G_{\gamma,\delta}$, that is the complete intersection of $\hat{\alpha}+1$ lines of type $(1,0)$ with $\hat{\beta}+1$ lines of type $(0,1)$. It is then clear that $H^0 \mathcal{I}_Z(\hat{\alpha}, \hat{\beta}) = H^0 \mathcal{I}_Z(\alpha, \beta) = 0$. In the second case we set $N = (\hat{\alpha}+1)(\hat{\beta}+1)$ $r = \hat{\alpha}+1$, $l = \beta+1$ and construct a graph g as in Lemma 3.2. Then we define $Z = \{P_{ij} : \{ij\} \in \text{Edges(g)}\}$. Then on any of the $\hat{\alpha}+1$ lines $L_1, \ldots, L_{\hat{\alpha}+1}$ there are $\hat{\beta}+1$ points of Z and on any of the $\beta+1$ lines $M_1, \ldots, M_{\beta+1}$ there are at least $N/(\beta+1) \geq \alpha+1$ points of Z. From this it is easy to see that $H^0 \mathcal{I}_Z(\hat{\alpha}, \hat{\beta}) = H^0 \mathcal{I}_Z(\alpha, \beta) = 0$. The third case is dealt similarly. $\qquad\square$

Avoiding the Brill-Noether locus

As in the preceding sections, we denote $d = \deg \mathscr{O}_C(\alpha, \beta) = \alpha b + \beta a$, $g = (a-1)(b-1)$ and assume $d \leq g-1$. Given G and Z as in Lemma 2.5 and Lemma 3.1, we consider the linear system S parametrizing the curves C of type (a, b) such that $G \setminus Z \subset C$. For any $\lambda \in S$ we denote by C_λ the corresponding curve. We also consider the incidence variety $\mathcal{C} = \{(x, \lambda) \in Q \times S : x \in C_\lambda\}$, which defines a flat family of curves over S by means of the second projection $p : \mathcal{C} \rightarrow S$. By Lemma 2.5, we know that there exists a smooth curve C_0 containing G such that $\mathscr{O}_{C_0}(a, -b) = \mathscr{O}_{C_0}$. Then on a suitable open affine neighborhood $0 \in B \subset S$, the pull-back $\mathcal{C}_B \rightarrow B$ is a flat family of smooth deformations of $C_0 = p^{-1}(0)$. We want to show that for a general $\lambda \in B$ and $C_\lambda = p^{-1}(\lambda)$ one has

$$H^0 \mathscr{O}_{C_\lambda}(\alpha + ma, \beta - mb) = 0 \quad m > 0.$$

For a smooth projective curve C one denotes $W_d(C)$ the Brill-Noether locus

$$W_d(C) = \{L \in \text{Pic}^d(C) : h^0(L) \neq 0\}$$

We want to prove that for a general $\lambda \in B$

$$\mathscr{O}_{C_\lambda}(\alpha + ma, \beta - mb) \notin W_d(C_\lambda).$$

By the general theory of the relative Picard scheme, see for example [4], one can associate to the family $\mathcal{C}_B$ a family

$$q : \mathcal{P}_d \rightarrow B,$$

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together with a universal line bundle $\mathcal{U}_d$ on $\mathcal{P}_d \times_B \mathcal{C}_B$, representing the functor on the category of algebraic schemes over B which associates to any $f : X \to B$ the set of equivalence classes of line bundles on $\mathcal{C}_X = X \times_B \mathcal{C}_B$ such that, for any closed point $x \in X$, the restriction $\mathcal{L} \otimes_{\mathscr{O}_B} \mathbb{C}(x)$ is a line bundle of degree d on $C_{f(x)}$, under the equivalence relation $\mathcal{L} \sim \mathcal{L} \otimes_{\mathscr{O}_B} f^*\mathcal{N}$ for any $\mathcal{N} \in \mathrm{Pic}(X)$. One has

$$q^{-1}(\lambda) \cong \mathrm{Pic}^d(C_\lambda)$$

for any closed point $\lambda \in B$. We denote $O \in \mathcal{P}_d$ the point corresponding to the line bundle $\mathscr{O}_{C_0}(\alpha, \beta)$ on C_0 and with $l : B \to \mathcal{P}_d$ the section of q defined by

$$l(\lambda) = \mathscr{O}_{C_\lambda}(\alpha + ma, \beta - mb) \in \mathrm{Pic}^d(C_\lambda). \tag{7}$$

The section l has the property that $l(0) = O \in \mathcal{P}_d$. Now we consider the d-th symmetric power

$$\mathcal{C}_B^{(d)} = \mathcal{C}_B \times_B \cdots \times_B \mathcal{C}_B / \Sigma_d$$

relative to B, with Σ_d the permutations groud on d letters, and the canonical map

$$u : \mathcal{C}_B^{(d)} \to \mathcal{P}_d$$

which restricts fiberwise to the Abel-Jacobi maps $u_\lambda : C_\lambda^{(d)} \to \mathrm{Pic}^d(C_\lambda)$. It is easy to see that $\mathcal{C}_B^{(d)}$ is smooth and that

$$u^{-1}(O) = u_0^{-1}(\mathscr{O}_{C_0}(\alpha, \beta)) \cong |\mathscr{O}_{C_0}(\alpha, \beta)| \subset C_0^{(d)}.$$

Then we define the subvariety

$$\mathcal{W} = u(\mathcal{C}_B^{(d)}) \subset \mathcal{P}_d.$$

As a set, one has $\mathcal{W} = \bigcup_\lambda W_d(C_\lambda)$. We want to show that $l(\lambda) \notin \mathcal{W}$ for some $\lambda \in B$. We denote by $\mathcal{T}_O(\mathcal{W})$ the tangent cone to $\mathcal{W}$ in the tangent space $T_O \mathcal{P}_d$. Our idea is to show that the image of the differential dl is not contained in $\mathcal{T}_O(\mathcal{W})$, i.e. there exists some $v \in T_0 B$ such that $dl(v) \notin \mathcal{T}_O(\mathcal{W})$. Indeed, if this is the case, then for a deformation C_λ of C_0 in the direction v one gets $l(\lambda) \notin \mathcal{W}$ for general λ.

Tangent cone to $\mathcal{W}$.

First of all, observe that by repeating the same construction of $\mathcal{P}_d$ in the case $d = 0$ one obtains the relative Picard group $\mathcal{P}_0 \to B$, with fibers $\mathrm{Pic}^0(C_\lambda)$. One obtains an isomorphism $\mathcal{P}_d \cong \mathcal{P}_0$, as schemes over B, by changing

the universal line bundle $\mathcal{U}_d$ to $\mathcal{U}_d \otimes q^* \mathscr{O}_{\mathcal{C}_B}(\alpha, \beta)^{-1}$. In other words, we can uniformly identify each $\operatorname{Pic}^d(C_\lambda)$ with the jacobian variety $\operatorname{Pic}^0(C_\lambda) \cong \mathcal{J}(C_\lambda)$ by sending $\mathscr{O}_{C_\lambda}(\alpha, \beta)$ to $0 \in \mathcal{J}(C_\lambda)$. We keep calling $\mathcal{W}$ the subvariety of $\mathcal{P}_0$ produced by means of the above identification, and $O \in \mathcal{P}_0$ the point corresponding to $\mathscr{O}_{C_0}(\alpha, \beta) \in \mathcal{P}_d$. Notice that $O \in \mathcal{P}_0$ corresponds to the origin $0 \in \mathcal{J}(C_0) \subset \mathcal{P}_0$.

We recall that by Serre duality and adjunction, one can identify the tangent spaces $T_0 \mathcal{J}(C_\lambda) = H^1(\Omega^1_{C_\lambda})$ with $H^0 \mathscr{O}_Q(a-2, b-2)^\vee$. So the tangent space $T_O \mathcal{P}_0$ is given by:

$$T_O \mathcal{P}_0 = H^0 \mathscr{O}_Q(a-2, b-2)^\vee \times T_0 B.$$

The following lemma describes the tangent cone to $\mathcal{W}$ in $T_O \mathcal{P}_0$

Lemma 3.3. *Let $\mathcal{T}$ be the tangent cone to $W_d(C_0)$ at the point $\mathscr{O}_{C_0}(\alpha, \beta)$. Then the following facts hold.*
1) The tangent cone $\mathcal{T}_O \mathcal{W}$ is supported on an irreducible closed set.
2) As a set, $\mathcal{T}_O \mathcal{W} = \mathcal{T} \times T_0 B$.

Proof. 1) It is easy to show that the map $u : \mathcal{C}_B^{(d)} \to \mathcal{W}$ is birational and it satisfies all the hypotheses of Lemma 1.1 chapter II of [1] and the subsequent corollary. The conclusion is that, as a set, $\mathcal{T}_O \mathcal{W}$ is given as the image of the normal bundle $\mathcal{N}$ to $u^{-1}(O)$ by means of the differential du. The fiber $u^{-1}(O)$ is the projective space $|\mathscr{O}_{C_0}(\alpha, \beta)|$, as observed above. Hence $\mathcal{T}_O \mathcal{W}$ is irreducible.
2) For any $\lambda \in B$, Kempf's theorem (see [1] p. 241) describes the tangent cone at $\mathscr{O}_{C_\lambda}(\alpha, \beta)$ of $W_d(C_\lambda) \subset \operatorname{Pic}^d(C_\lambda)$ as the affine cone

$$\bigcup_{\sigma \in H^0 \mathscr{O}_{C_\lambda}(\alpha, \beta)} \mu(\sigma \otimes H^0 \mathscr{O}_{C_\lambda}(\hat{\alpha}, \hat{\beta}))^\perp \subset H^0(\mathscr{O}_{C_\lambda}(a-2, b-2))^\vee,$$

where $\mu : H^0 \mathscr{O}_{C_\lambda}(\alpha, \beta) \otimes H^0 \mathscr{O}_{C_\lambda}(\hat{\alpha}, \hat{\beta}) \to H^0 \mathscr{O}_{C_\lambda}(a-2, b-2)$ is the multiplication map of global sections. Recall that one may also identify $H^0 \mathscr{O}_{C_\lambda}(\alpha, \beta) \cong H^0 \mathscr{O}_Q(\alpha, \beta)$ and $H^0 \mathscr{O}_{C_\lambda}(\hat{\alpha}, \hat{\beta}) \cong H^0 \mathscr{O}_Q(\hat{\alpha}, \hat{\beta})$ and μ is identified with the analogous multiplication map on Q

$$\mu_Q : H^0 \mathscr{O}_Q(\alpha, \beta) \otimes H^0 \mathscr{O}_Q(\hat{\alpha}, \hat{\beta}) \to H^0 \mathscr{O}_Q(a-2, b-2).$$

As a result, we see that the tangent cones at $\mathscr{O}_{C_\lambda}(\alpha, \beta)$ of $W_d(C_\lambda)$ are all the same cone

$$\mathcal{T} = \bigcup_{\sigma \in H^0 \mathscr{O}_Q(\alpha, \beta)} \mu(\sigma \otimes H^0 \mathscr{O}_Q(\hat{\alpha}, \hat{\beta}))^\perp$$

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inside the space $H^0\mathcal{O}_Q(a-2, b-2)^\vee$. By the identifications $\mathrm{Pic}^d(C_\lambda) \cong J(C_\lambda)$ which map $\mathcal{O}_{C_\lambda}(\alpha, \beta)$ to $0 \in J(C_\lambda)$, we see that the tangent cones to $\mathcal{W}(\lambda)$ inside $H^0\mathcal{O}_Q(a-2, b-2) \cong T_0 J(C_\lambda)$ are all equal to $\mathcal{T}$. It follows that the tangent cone $\mathcal{T}_O\mathcal{W}$ contains the product cone $\mathcal{T} \times T_0 B$. By 1) and the equality of dimensions, we are done. $\qquad\square$

Computation of dl

Let l be the section of $\mathcal{P}_d \to B$ defined in (7). Recall that for any curve C the Picard group $\mathrm{Pic}^0(C)$ is identified with the jacobian variety $J(C)$ by means of the Abel-Jacobi map

$$D = \sum_i (P_i - Q_i) \mapsto \sum_i \left(\int_{Q_i}^{P_i} \omega_1, \ldots, \int_{Q_i}^{P_i} \omega_g \right),$$

with $\omega_1, \ldots, \omega_g$ a basis of $H^0(\Omega_C^1)$, and integrals defined along arbitrary paths in C from Q_i to P_i.

Now let C_t be a sub-family of $\mathcal{C}_B \to B$ parametrized by a line $t \mapsto tv \in B$, for a given tangent vector $v \in T_0 B$. We know that

$$l(tv) = \mathcal{O}_{C_t}(ma, -mb) = \mathcal{O}_{C_t}(mX_t - mY_t),$$

with $X_t = \sum_{i=1}^N P_i(t)$ and $Y_t = \sum_{i=1}^N Q_i(t)$, $N = \deg Z$ and $P_i(0) = Q_i(0) \in Z$ for $i = 1, \ldots, N$, hence in particular $X_0 = Y_0 = Z$. Now let us fix our attention on one point $P = P_i(0) = Q_i(0) \in Z$ and assume that the curves C_t have equations $f(u, v, t) = 0$ in affine coordinates u, v centered at P. We may assume without loss of generality that the partial derivative f_u does not vanish in a neighborhood of $(u, v, t) = (0, 0, 0)$. Consider a basis of rational functions of $H^0\mathcal{O}_Q(a-2, b-2)$ given by $g = (a-1)(b-1)$ linearly independent rational functions $h_1(u, v), \ldots, h_g(u, v)$ such that $\mathrm{div}(h_j) + (a-2)L + (b-2)M \geq 0$ for any $j = 0, \ldots, g$, with L the line with equation $u = 0$ and M the line with equation $v = 0$. Then we know that for any t in a neighborhood of $t = 0$ the differential forms $(\omega_1, \ldots, \omega_g) = ((h_1/f_u)dv, \ldots, (h_g/f_u)dv)$ are a basis of $H^0\Omega_{C_t}^1$. For any fixed index i one can consider the Abel-Jacobi map

$$t \mapsto \left(\int_{Q_i(t)}^{P_i(t)} \omega_1, \ldots, \int_{Q_i(t)}^{P_i(t)} \omega_g \right) = s_i(t).$$

Up to suitably shrinking B, one can uniformly choose the integration paths γ_t for any $t \in B$ in the following way. We can assume $\gamma_t \subset C_t$ defined by a function $\gamma_t(s) = \gamma(s, t) = (u(s, t), v(s, t))$, with $(s, t) \in [0, 1] \times B$, satisfying the following conditions: $\gamma(s, t)$ is a $\mathcal{C}^\infty$ function, $\gamma(s, 0) = P$ for any $s \in [0, 1]$ and $f(\gamma(s, t), t) \equiv 0$. Then we will use the following formula.

Lemma 3.4. *Let $a(u, v, t)$ be a $\mathcal{C}^1$ function and $\gamma(s, t)$ a family of paths satisfying the assumptions above. Then*

$$\frac{d}{dt} \int_{\gamma_t} a \, dv \bigg|_{t=0} = a(0, 0, 0)(v_t(1, 0) - v_t(0, 0)).$$

Proof.

$$\frac{d}{dt} \int_{\gamma_t} a \, dv \bigg|_{t=0} = \frac{d}{dt} \int_0^1 a v_s \, ds \bigg|_{t=0} = \int_0^1 ((a_u u_t + a_v v_t + a_t) v_s + a v_{st}) \, ds \bigg|_{t=0}.$$

Now, since $\gamma(s, 0) = (u(s, 0), v(s, 0)) \equiv (0, 0)$ and hence $v_s(s, 0) \equiv 0$, we get

$$\frac{d}{dt} \int_{\gamma_t} a \, dv \bigg|_{t=0} = \int_0^1 a(\gamma(s, 0), 0) v_{st}(s, 0)) \, ds = a(0, 0, 0)(v_t(1, 0) - v_t(0, 0)).$$

$\square$

We apply this result to

$$l(tv) = \sum_{i=1}^N m \left(\int_{Q_i(t)}^{P_i(t)} \omega_1, \dots, \int_{Q_i(t)}^{P_i(t)} \omega_g \right),$$

writing ω_i in the form $a \, dv$ for any $i = 1, \dots, g$. We get

$$\frac{dl(tv)}{dt} \bigg|_{t=0} = m \sum_{i=1}^N \lambda_i(\omega_1(P_i), \dots, \omega_g(P_i)) = m \sum_{i=1}^N \lambda_i \Phi(P_i),$$

with Φ representing the canonical map $\phi : C_0 \to \mathbb{P}^{g-1}$ and the coefficient λ_i equal to the v component of $P_i'(t) - Q_i'(t)$ for any $i = 1, \dots, N$.

Claim: One can choose the vector $v \in T_0 B$ and hence the family C_t in such a way that all λ_i's above are non-zero.

Indeed we know that the $P_i(t)$ belong to $(1, 0)$ lines and the $Q_i(t)$ belong to $(0, 1)$ lines. So in suitable affine coordinates (u, v) one sees that $Q_i(t) = (u(t), c)$ with c some constant. Hence the v component of $Q_i'(t)$ is zero, so the only contribution to λ_i comes from $P_i'(t)$. In the family of curves $\mathcal{C}_B$ one can find a subfamily of curves C_t, with t in a suitable neighborhood of $0 \in \mathbb{C}$, passing through the points $P_i(t) = P_i + (c_i, t)$, for any $i = 1, \dots, N$, with $c_i = u(P_i)$ and $\{P_1, \dots, P_N\} = Z$. This is possible since the dimension of the linear system of curves S passing through $G \setminus Z$, of which B is an open set, is $ab + a + b - (ab - N) = a + b + N > N$. Hence we find $\lambda_i = 1$ for any $i = 1, \dots, N$, proving the claim.

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Conclusion of the proof. Assume we have a family C_t as above, with tangent vector $v \in T_0 B$. Then $dl(v) \in T_O \mathcal{P}_0 = H^0 \mathcal{O}_Q(a-2, b-2)^* \times T_0 B$ has the form

$$dl(v) = \left(m \sum_{i=1}^{N} \lambda_i \Phi(P_i), v \right).$$

By the discussion above, we see that $dl(v) \in \mathcal{T}_O \mathcal{W}$ if and only if

$$\sum_{i=1}^{N} \lambda_i \Phi(P_i) \in \mathcal{T}.$$

Assume that this fact holds. Then there exists $\sigma \in H^0 \mathcal{O}_Q(\alpha, \beta)$ such that for any $\tau \in H^0 \mathcal{O}_Q(\hat{\alpha}, \hat{\beta})$, the hyperplane of $\mathbb{P} H^0(a-2, b-2)^\vee$ represented by $\sigma\tau \in H^0 \mathcal{O}_Q(a-2, b-2)$ vanishes on $R = \sum_{i=1}^{N} \lambda_i \Phi(P_i)$. But now recall that the set $Z = \{P_1, \ldots, P_N\}$ imposes independent conditions to $H^0 \mathcal{O}_Q(\hat{\alpha}, \hat{\beta})$, so for any $i = 1, \ldots, N$ one finds a non-zero form $\tau_i \in H^0 \mathcal{O}_Q(\hat{\alpha}, \hat{\beta})$ such that $\tau_i(P_i) \neq 0$ and $\tau_i(P_j) = 0$ for any $j \neq i$. So we have

$$0 = (\sigma\tau_i)(R) = \lambda_i \sigma(P_i)\tau_i(P_i),$$

from which it follows $\sigma(P_i) = 0$. Hence σ vanishes on Z, which contradicts the fact that $H^0 \mathcal{I}_Z(\alpha, \beta) = 0$. $\qquad\square$

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